

EQUALITY OF UNIFORM AND CARLEMAN SPECTRA FOR BOUNDED MEASURABLE FUNCTIONS

BOLIS BASIT AND ALAN J. PRYDE

ABSTRACT. In this paper we study various types of spectra of functions $\phi : \mathbb{J} \rightarrow X$, where $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$ and X is a complex Banach space. We show that uniform spectrum defined in [15] coincides with Carleman spectrum for $\phi \in L^\infty(\mathbb{R}, X)$. This result holds true also for Laplace (half-line) spectrum for $\phi \in L^\infty(\mathbb{R}_+, X)$. We also indicate a class of bounded measurable functions for which Laplace spectrum and Carleman spectrum are equal.

§0. INTRODUCTION

In this paper we study various types of spectra of functions $\phi : \mathbb{J} \rightarrow X$, where $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$ and X is a complex Banach space. If $\phi \in BUC(\mathbb{J}, X)$, the space of bounded uniformly continuous functions, it is easily verified using Bochner integration that the Laplace transform $\mathcal{L}\phi : \mathbb{C}_+ \rightarrow X$, where $\mathcal{L}\phi(\lambda) = \int_0^\infty e^{-\lambda t} \phi(t) dt$, has a holomorphic extension in a neighbourhood of some point $i\omega \in i\mathbb{R}$ if and only if $\mathcal{L}_u\phi : \mathbb{C}_+ \rightarrow BUC(\mathbb{J}, X)$, where $\mathcal{L}_u\phi(\lambda)(s) = \int_0^\infty e^{-\lambda t} \phi(t+s) dt$ for $s \in \mathbb{J}$, has a holomorphic extension in a neighbourhood of $i\omega$. The Laplace spectrum is $sp^{\mathcal{L}}(\phi) := \{\omega \in \mathbb{R} : i\omega \text{ is a singular point for } \mathcal{L}\phi\}$ and the uniform Laplace spectrum is $sp^{\mathcal{L}_u}(\phi) := \{\omega \in \mathbb{R} : i\omega \text{ is a singular point for } \mathcal{L}_u\phi\}$. The Carleman transform (see (1.3)), spectrum (see (1.7)) and the uniform spectrum are defined similarly. In [15], [24], the uniform Carleman spectrum $sp^{\mathcal{C}_u}(\phi)$ for $\phi \in BC(\mathbb{R}, X)$ is introduced and it is shown that $sp^{\mathcal{C}}(\phi) \subset sp^{\mathcal{C}_u}(\phi)$. Many properties of $sp^{\mathcal{C}}(\phi)$ are shown to hold true for $sp^{\mathcal{C}_u}(\phi)$ (see [15, Proposition 2.3]); however, equality is not established.

In §1 we collect notation and definitions and prove some preliminaries. In particular, we prove in Proposition 1.5 a useful necessary condition for a point ω to be in the complement of the reduced Beurling spectrum $sp_{C_0(\mathbb{J}, X)}(\phi)$.

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In §2, we establish some tools which enable us to calculate the Laplace and weak Laplace spectra and relate them to the reduced Beurling spectrum relative to a class \mathcal{A} satisfying (1.12). Using a new property (Theorem 2.4(i)) of the weak Laplace spectrum we give simple proofs of several tauberian results of Ingham [20], [22] and their generalizations by Chill and others (see [2], [3], [4, 4.10, p. 332], [11] and references therein).

In §3, we study bounded C_0 -semigroups $T(t)$, $t \in \mathbb{J}$ with generator A . We give proofs of the identities $i \operatorname{sp}^{\mathcal{L}}(T(\cdot)) = \sigma(A) \cap i\mathbb{R}$ and, when $\mathbb{J} = \mathbb{R}$, of the equality of the Arveson, Beurling and Carleman spectra of both the orbits $T(\cdot)x$ and groups $T(\cdot)$ (see (3.6), (3.7)); in particular, in Corollary 3.3 we show that Arveson spectrum $\operatorname{sp}^A(T(\cdot)x)$ is equal $\sigma_u(A, x)$, the local unitary spectrum of A at x defined in [10, §3]. Since isometric semigroups on \mathbb{R}_+ can be extended uniquely to isometric groups on \mathbb{R} , Corollary 3.3 extends [10, Theorem 2.2].

In §4, we prove $\operatorname{sp}^{\mathcal{L}_u}(\phi) = \operatorname{sp}^{\mathcal{L}}(\phi)$ and, when $\mathbb{J} = \mathbb{R}$, $\operatorname{sp}^{\mathcal{L}_u}(\phi) = \operatorname{sp}^{\mathcal{L}}(\phi)$ for $\phi \in L^\infty(\mathbb{J}, X)$. These results seem new for $BC(\mathbb{R}, X)$ and the proofs are new even for $BUC(\mathbb{R}, X)$.

Finally, in section §5, we indicate a subclass of $L^\infty(\mathbb{R}, X)$ for which the Laplace spectrum coincides with the Carleman spectrum. This class includes almost periodic, almost automorphic, Levitan almost periodic and recurrent functions.

§1. NOTATION, DEFINITIONS AND PRELIMINARIES

In the following $\mathbb{R}_+ = [0, \infty)$, $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ and $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$. Denote by X a complex Banach space. If Y, Z are locally convex topological spaces, $L(Y, Z)$ will denote the space of all bounded linear operators from Y to Z and $L(Y) = L(Y, Y)$. The Schwarz space of rapidly decreasing functions is denoted by $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R}, X) = L(\mathcal{S}(\mathbb{R}), X)$ (see [31]) is the space of X -valued tempered distributions on \mathbb{R} . The action of an element $S \in \mathcal{S}'(\mathbb{R}, X)$ on $f \in \mathcal{S}(\mathbb{R})$ is denoted $\langle S, f \rangle$. If ϕ is a X -valued function defined on \mathbb{J} , then $\phi_s, \Delta_s \phi$ will stand for functions defined on \mathbb{J} by $\phi_s(t) = \phi(t + s)$, $\Delta_s \phi(t) = \phi_s(t) - \phi(t)$ for all $s \in \mathbb{J}$, $|\phi|$ will denote the function $|\phi|(t) := \|\phi(t)\|$ for all $t \in \mathbb{J}$ and $\|\phi\|_\infty := \sup_{t \in \mathbb{J}} \|\phi(t)\|$. If $\phi \in L^1_{loc}(\mathbb{J}, X)$, then $P\phi$

and $M_h\phi$ will denote the indefinite integral and mollifier of ϕ defined respectively by $P\phi(t) = \int_0^t \phi(s) ds$ and $M_h\phi(t) = (1/h) \int_0^h \phi(t+s) ds$ for $h > 0$. For $f \in L^1(\mathbb{R}, \mathbb{C})$ and $\phi \in L^\infty(\mathbb{R}, X)$ or $f \in L^1(\mathbb{R}, X)$ and $\phi \in L^\infty(\mathbb{R}, \mathbb{C})$ the Fourier transform \hat{f} and convolution $\phi * f$ are defined respectively by $\hat{f}(\omega) = \int_{-\infty}^{\infty} \gamma_{-\omega}(t) f(t) dt$ and $\phi * f(t) = \int_{-\infty}^{\infty} \phi(t-s) f(s) ds$, where $\gamma_\omega(t) = e^{i\omega t}$. The Fourier transform of $S \in \mathcal{S}'(\mathbb{R}, X)$ is the tempered distribution \hat{S} defined by $\langle \hat{S}, f \rangle = \langle S, \hat{f} \rangle$ for all $f \in \mathcal{S}(\mathbb{R})$. All integrals are Lebesgue-Bochner integrals (see [4, pp. 6-15], [16, p. 318], [19, p. 76]).

We recall [19, Definition 3.5.5, p. 74] that an operator valued function $F : \mathbb{J} \rightarrow L(X)$ is strongly measurable (strongly integrable) if $F(\cdot)x$ is measurable (integrable) for each $x \in X$. We denote by $L_s^\infty(\mathbb{J}, L(X))$ the Banach space of all (essentially) bounded strongly measurable operator-valued functions F such that $|F(\cdot)| \in L^\infty(\mathbb{J})$. Since $F(\cdot)x \in L^\infty(\mathbb{J}, X)$ for each $x \in X$ and $a \in \mathbb{J}$ we may define the *strong integral* of F by $(\int_0^a F(t) dt)x := \int_0^a F(t)x dt$. Similarly for $h > 0$ and $t \in \mathbb{J}$, the *strong mollifier* $M_h F(t) \in L(X)$ is defined by $(M_h F(t))x = M_h(F(t)x) = (1/h) \int_0^h F(t+s)x ds$. For example if $T(t) \in L(X)$ for $t \in \mathbb{J}$ is a bounded C_0 -semigroup, then by [16, p 616], $|T(\cdot)| \in L^\infty(\mathbb{J})$ and so $T(\cdot) \in L_s^\infty(\mathbb{J}, L(X))$. In particular, the translation semi-groups $S^\mathbb{J}(t) : BUC(\mathbb{J}, X) \rightarrow BUC(\mathbb{J}, X)$ defined by $S^\mathbb{J}(t)\phi = \phi_t$ for $t \in \mathbb{J}$ are strongly continuous. Hence $S^\mathbb{J}(\cdot) \in L_s^\infty(\mathbb{J}, L(BUC(\mathbb{J}, X)))$. For simplicity we write sometimes $S(\cdot) = S^\mathbb{R}(\cdot)$ and $S^+(\cdot) = S^{\mathbb{R}_+}(\cdot)$.

In this paper we consider the space $L_{loc}^1(\mathbb{R}_+, X)$ as a subspace of $L_{loc}^1(\mathbb{R}, X)$ by identifying a function ϕ defined on \mathbb{R}_+ with its extension by 0 to \mathbb{R} .

If $\phi \in L_{loc}^1(\mathbb{J}, X) \cap \mathcal{S}'(\mathbb{R}, X)$, then

$$f\phi|_{\mathbb{R}_+}, e_\lambda \phi|_{\mathbb{R}_+} \in L^1(\mathbb{R}_+, X) \text{ for } f \in \mathcal{S}(\mathbb{R}) \text{ and } e_\lambda(t) = e^{-\lambda t}, \lambda \in \mathbb{C}_+,$$

and the *Laplace transform* $\mathcal{L}\phi$ is defined by

$$(1.1) \quad \mathcal{L}\phi(\lambda) = \int_0^\infty e^{-\lambda t} \phi(t) dt \quad \text{for } \lambda \in \mathbb{C}_+.$$

Clearly, $\mathcal{L}\phi$ is holomorphic on \mathbb{C}_+ . In particular, if $F \in L_s^\infty(\mathbb{J}, L(X))$ then $\mathcal{L}F(\cdot)x$ is holomorphic on \mathbb{C}_+ for each $x \in X$. For $\lambda \in \mathbb{C}_+$ we define $\mathcal{L}F(\lambda)$ by

$$(1.2) \quad \mathcal{L}F(\lambda)x = \mathcal{L}F(\cdot)x(\lambda).$$

Denote the *Carleman transform* of $\phi \in L_{loc}^1(\mathbb{R}, X) \cap \mathcal{S}'(\mathbb{R}, X)$ by

$$(1.3) \quad \mathcal{C}\phi(\lambda) = \begin{cases} \mathcal{L}^+\phi(\lambda) = \int_0^\infty e^{-\lambda t} \phi(t) dt, & \text{if } \lambda \in \mathbb{C}_+ \\ \mathcal{L}^-\phi(\lambda) = -\int_0^\infty e^{\lambda t} \phi(-t) dt, & \text{if } \lambda \in \mathbb{C}_-. \end{cases}$$

Then $\mathcal{C}\phi$ is an X -valued function which is holomorphic on $\mathbb{C} \setminus i\mathbb{R}$.

Similarly, since the Carleman transform $\mathcal{C}F(\cdot)x$ of $F \in L_s^\infty(\mathbb{R}, L(X))$ is holomorphic on $\mathbb{C} \setminus i\mathbb{R} = \mathbb{C}_+ \cup \mathbb{C}_-$ for each $x \in X$ we define $\mathcal{C}F(\lambda)$ by

$$(1.4) \quad \mathcal{C}F(\lambda)x = \mathcal{C}F(\cdot)x(\lambda).$$

It is easily verified that $\mathcal{L}F(\lambda) \in L(X)$ for $\lambda \in \mathbb{C}_+$ and $\mathcal{C}F(\lambda) \in L(X)$ for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. Since $\mathcal{L}F(\cdot)x$ (respectively $\mathcal{C}F(\cdot)(\lambda)x$) is holomorphic on \mathbb{C}_+ ($\mathbb{C} \setminus i\mathbb{R}$) for each $x \in X$, it follows that $\mathcal{L}F$ (respectively $\mathcal{C}F$) is holomorphic on \mathbb{C}_+ ($\mathbb{C} \setminus i\mathbb{R}$), by [19, Theorem 3.10.1, p. 93].

If $\phi \in L^1(\mathbb{J}, X)$, then $\mathcal{L}\phi$ has a continuous extension to $\overline{\mathbb{C}_+}$ given also by (1.1). By the Riemann-Lebesgue lemma, if $g(\omega) = \mathcal{L}\phi(i\omega)$, then $g \in C_0(\mathbb{R}, X)$. Note that $g = \widehat{\phi|_{\mathbb{R}_+}}$ is the Fourier transform of $\phi|_{\mathbb{R}_+}$ extended by 0 to \mathbb{R} . Moreover, if $\phi \in \mathcal{S}(\mathbb{J}, X)$, then $g \in C_0^\infty(\mathbb{R}, X)$. If $\phi \in L^1(\mathbb{R}, X)$, then $\mathcal{C}\phi$ has a continuous extension to \mathbb{C} and the Fourier transform $\hat{\phi} = \mathcal{C}\phi(i\cdot)$ satisfies $\hat{\phi} \in C_0(\mathbb{R}, X)$. If $f \in \mathcal{S}(\mathbb{R}, X)$, then $\hat{f} \in \mathcal{S}(\mathbb{R}, X)$ by [32, p. 146] for $X = \mathbb{C}$. It follows that if $\phi \in L_{loc}^1(\mathbb{R}_+, X) \cap \mathcal{S}'(\mathbb{R}, X)$, then $\hat{\phi} \in \mathcal{S}'(\mathbb{R}, X)$, by [32, p. 151] for $X = \mathbb{C}$. So, if $e_a(t) = e^{-at}$, then $\mathcal{L}\phi(a + i\cdot) = \widehat{e_a\phi}$ is an $\mathcal{S}'(\mathbb{R}, X)$ -valued function for all $a > 0$. Moreover, if $f \in \mathcal{S}(\mathbb{R})$, $\langle \mathcal{L}\phi(a + i\cdot), f \rangle = \langle \widehat{e_a\phi}, f \rangle = \langle e_a\phi, \hat{f} \rangle \rightarrow \langle \phi, \hat{f} \rangle = \langle \hat{\phi}, f \rangle$, where the limit exists as $a \searrow 0$ by the Lebesgue dominating convergence theorem. This means for $\phi \in L_{loc}^1(\mathbb{R}_+, X) \cap \mathcal{S}'(\mathbb{R}, X)$,

$$(1.5) \quad \lim_{a \searrow 0} \mathcal{L}\phi(a + i\cdot) = \hat{\phi} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}, X).$$

For a holomorphic function $\zeta : \Sigma \rightarrow X$, where $\Sigma = \mathbb{C}_+$ or $\Sigma = \mathbb{C} \setminus i\mathbb{R}$, the point $i\omega \in i\mathbb{R}$ is called a *regular point* for ζ or ζ is called *holomorphic* at $i\omega$, if ζ has an extension $\bar{\zeta}$ which is holomorphic in a neighbourhood $V \subset \mathbb{C}$ of $i\omega$.

Points $i\omega$ which are *not regular points* are called *singular points*.

The *Laplace spectrum* of $\Phi \in L_{loc}^1(\mathbb{J}, X) \cap \mathcal{S}'(\mathbb{R}, X)$ or $\Phi \in L_s^\infty(\mathbb{J}, L(X))$ is defined by

$$(1.6) \quad sp^{\mathcal{L}}(\Phi) := \{\omega \in \mathbb{R} : i\omega \text{ is a singular point for } \mathcal{L}\Phi\}. \text{ See [4, p. 275].}$$

The Laplace spectrum is called also the half-line spectrum (see [4, p. 275]).

The *Carleman spectrum* of $\Phi \in L_{loc}^1(\mathbb{R}, X) \cap \mathcal{S}'(\mathbb{R}, X)$ or $\Phi \in L_s^\infty(\mathbb{R}, L(X))$ is defined by

$$(1.7) \quad sp^{\mathcal{C}}(\Phi) := \{\omega \in \mathbb{R} : i\omega \text{ is a singular point for } \mathcal{C}\Phi\}. \text{ See [4, (4.26)]}.$$

We note that for $\phi \in L^\infty(\mathbb{R}^+, X)$, $\omega \notin sp^{\mathcal{L}}(\phi)$ respectively $\phi \in L^\infty(\mathbb{R}, X)$, $\omega \notin sp^{\mathcal{C}}(\phi)$, one has

$$(1.8) \quad \lim_{\lambda \rightarrow 0} \mathcal{L}(\gamma_{-\omega}\phi)(\lambda) = \overline{\mathcal{L}}(\gamma_{-\omega}\phi)(0) \text{ respectively} \\ \lim_{\lambda \rightarrow 0} \mathcal{C}(\gamma_{-\omega}\phi)(\lambda) = \overline{\mathcal{C}}(\gamma_{-\omega}\phi)(0).$$

Indeed, by the definitions that $0 \notin sp^{\mathcal{L}}(\gamma_{-\omega}\phi)$ or $0 \notin sp^{\mathcal{C}}(\gamma_{-\omega}\phi)$, it follows $\mathcal{L}(\gamma_{-\omega}\phi)$ or $\mathcal{C}(\gamma_{-\omega}\phi)$ has a holomorphic extension $\overline{\mathcal{L}}(\gamma_{-\omega}\phi)$ or $\overline{\mathcal{C}}(\gamma_{-\omega}\phi)$ in a neighbourhood of 0.

For a holomorphic function $\zeta : \mathbb{C}_+ \rightarrow X$, the point $i\omega \in i\mathbb{R}$ is called a *weak regular point* for ζ if there exist $\varepsilon > 0$ and $h \in L^1(\omega - \varepsilon, \omega + \varepsilon)$ such that

$$(1.9) \quad \lim_{a \searrow 0} \int_{-\infty}^{\infty} \zeta(a + is)\varphi(s) ds = \int_{\omega - \varepsilon}^{\omega + \varepsilon} h(s)\varphi(s) ds \\ \text{for all } \varphi \in \mathcal{D}(\mathbb{R}) \text{ with } \text{supp } \varphi \subset]\omega - \varepsilon, \omega + \varepsilon[.$$

Points $i\omega$ which are not weak regular points are called *weak singular points*.

The *weak Laplace spectrum* of $\Phi \in L_{loc}^1(\mathbb{J}, X) \cap \mathcal{S}'(\mathbb{R}, X)$ or $\Phi \in L_s^\infty(\mathbb{J}, L(X))$ is defined by (see [4, p. 324])

$$(1.10) \quad sp^{w\mathcal{L}}(\Phi) := \{\omega \in \mathbb{R} : i\omega \text{ is not a weak regular point for } \mathcal{L}\Phi\}.$$

The definitions imply

$$(1.11) \quad sp^{w\mathcal{L}}(\Phi) \subset sp^{\mathcal{L}}(\Phi) \subset sp^{\mathcal{C}}(\Phi)$$

and $sp^{w\mathcal{L}}(\Phi) = \emptyset$ if $\Phi \in L^1(\mathbb{R}, X)$.

We conclude this section by recalling (see [7, p. 117]) that a subset $\mathcal{A} \subset L^\infty(\mathbb{J}, X)$ is called *BUC-invariant* if for each $\psi \in BUC(\mathbb{R}, X)$ with $\psi|_{\mathbb{J}} \in \mathcal{A}$, one has $\psi_a|_{\mathbb{J}} \in \mathcal{A}$ for each $a \in \mathbb{R}$.

In the following we assume

$$(1.12) \quad \mathcal{A} \text{ is a } BUC\text{-invariant (closed) subspace of } L^\infty(\mathbb{J}, X).$$

Examples of such \mathcal{A} include

$$\{0\}, C_0(\mathbb{J}, X), AP(\mathbb{R}, X), AAP(\mathbb{J}, X),$$

the spaces consisting respectively of the zero function (when $\mathbb{J} = \mathbb{R}$), all continuous vanishing at infinity, almost periodic (when $\mathbb{J} = \mathbb{R}$) and asymptotically almost periodic functions. Such an \mathcal{A} is called a Λ -class if additionally $\mathcal{A} \subset BUC(\mathbb{J}, X)$, contains all constants and is also closed under multiplication by characters.

A point $\omega \in \mathbb{R}$ is called \mathcal{A} -regular for $\phi \in L_{loc}^1(\mathbb{J}, X) \cap \mathcal{S}'(\mathbb{R}, X)$, where \mathcal{A} is a

class satisfying (1.12) if there is $f \in \mathcal{S}(\mathbb{R})$ such that $\hat{f}(\omega) \neq 0$ and $\phi * f|_{\mathbb{J}} \in \mathcal{A}$. The *reduced Beurling spectrum* of ϕ with respect to \mathcal{A} is defined by (see [5, (4.1.1)], [6, (2.9)], [14] and references therein)

$$(1.13) \quad sp_{\mathcal{A}}(\phi) := \{\omega \in \mathbb{R} : \omega \text{ is not an } \mathcal{A}\text{-regular point for } \phi\}.$$

We note that if $\phi \in L^\infty(\mathbb{J}, X)$, then a point $\omega \in \mathbb{R}$ is \mathcal{A} -regular for ϕ if and only if there is $f \in L^1(\mathbb{R})$ such that $\hat{f}(\omega) \neq 0$ and $\phi * f|_{\mathbb{J}} \in \mathcal{A}$.

If $\mathcal{A} = \{0\}$, then $sp_{\{0\}}(\phi)$ is just the Beurling spectrum defined in (1.17) below. We recall the following property of $sp_{\mathcal{A}}(\phi)$. For $\phi \in L^1_{loc}(\mathbb{R}, X) \cap \mathcal{S}'(\mathbb{R}, X)$, $f \in \mathcal{S}(\mathbb{R})$,

$$(1.14) \quad sp_{\mathcal{A}}(\phi * f) \subset sp_{\mathcal{A}}(\phi) \cap \text{supp } \hat{f}.$$

For the proof, see the references in [8, Proposition 1.1 (ii)] and, for the case $\mathcal{A} = \{0\}$, [26, Proposition 06(i), p. 25].

We recall (see [7], [9], [29], [30]) that a function $\phi \in L^1_{loc}(\mathbb{J}, X)$ is called *ergodic* if there is a constant $m(\phi) \in X$ such that

$$\sup_{x \in \mathbb{J}} \left\| \frac{1}{T} \int_0^T \phi(t+s) ds - m(\phi) \right\| \rightarrow 0 \text{ as } T \rightarrow \infty.$$

The limit $m(\phi)$ is called the *mean* of ϕ . The set of all such ergodic functions will be denoted by $\mathcal{E}(\mathbb{J}, X)$. We set $\mathcal{E}_b(\mathbb{J}, X) = \mathcal{E}(\mathbb{J}, X) \cap L^\infty(\mathbb{J}, X)$, $\mathcal{E}_0(\mathbb{J}, X) = \{\phi \in \mathcal{E}_b(\mathbb{J}, X) : m(\phi) = 0\}$, $\mathcal{E}_{ub}(\mathbb{J}, X) = \mathcal{E}(\mathbb{J}, X) \cap BUC(\mathbb{J}, X)$ and $\mathcal{E}_{u,0}(\mathbb{J}, X) = \mathcal{E}_{ub}(\mathbb{J}, X) \cap \mathcal{E}_0(\mathbb{J}, X)$.

Simple calculations show that for $\phi \in L^\infty(\mathbb{J}, X)$, $t \in \mathbb{J}$, $h, T > 0$, one has

$$\begin{aligned} \frac{1}{T} \int_0^T [\phi(t+s) - \phi(t+s+h)] ds &= \frac{1}{T} \int_0^h [\phi(t+s) - \phi(t+s+T)] ds, \\ \frac{1}{T} \int_0^T [M_h \phi(t+s) - \phi(t+s)] ds &= \frac{1}{hT} \int_0^h \int_0^u [\phi(t+s) - \phi(t+s+T)] ds du, \end{aligned}$$

which implies

$$(1.15) \quad \phi - \phi_h \text{ and } \phi - M_h \phi \in \mathcal{E}_0(\mathbb{J}, X).$$

In the sequel we need the following analogue of Wiener's theorem on Fourier series. See [12, Proposition 1.1.5 (b), p. 22].

Proposition 1.1. *Let $f \in L^1(\mathbb{R})$ and $\hat{f}(\omega) \neq 0$ for all $\omega \in K$ a compact set. Then there is $g \in L^1(\mathbb{R})$ such that $\hat{f}(\omega) \cdot \hat{g}(\omega) = 1$ for all $\omega \in K$.*

For $\phi \in L^\infty(\mathbb{R}, X)$, the corresponding Arveson spectrum, Beurling spectrum and maximal ideal (see [14], [27, p. 184]) are defined respectively by

$$(1.16) \quad sp^A(\phi) = \{\lambda \in \mathbb{R} : \text{for each } \varepsilon > 0$$

there exists $f \in L^1(\mathbb{R})$, $\text{supp } \hat{f} \subset]\lambda - \varepsilon, \lambda + \varepsilon[$ and $f * \phi \neq 0$

$$(1.17) \quad sp^B(\phi) = \{\lambda \in \mathbb{R} : \text{if } f \in L^1(\mathbb{R}), \hat{f}(\lambda) = 1 \text{ then } f * \phi \neq 0\},$$

$$(1.18) \quad I(\phi) = \{f \in L^1(\mathbb{R}) : \phi * f = 0\}.$$

The following result is well known and relates (1.17) to [4, p. 321], [5, Definition 4.1.2]. We include a proof for the benefit of the reader.

Proposition 1.2. *For $\phi \in L^\infty(\mathbb{R}, X)$, one has*

$$(1.19) \quad sp^B(\phi) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) = 0 \text{ for all } f \in I(\phi)\} = sp^A(\phi).$$

Proof. The first equality of (1.19) is easily verified. Now, let $\lambda \notin sp^B(\phi)$. There are $f_0 \in L^1(\mathbb{R})$ with $\hat{f}_0(\lambda) = 1$ but $f_0 * \phi = 0$; $\delta > 0$ such that $|\hat{f}_0(\mu)| \geq 1/2$ for $\mu \in [\lambda - \delta, \lambda + \delta]$; and by Proposition 1.1, $g_0 \in L^1(\mathbb{R})$ such that $\hat{g}_0(\mu) \cdot \hat{f}_0(\mu) = 1$ for $\mu \in [\lambda - \delta, \lambda + \delta]$. Let $f \in L^1(\mathbb{R})$ and $\text{supp } \hat{f} \subset [\lambda - \delta, \lambda + \delta]$. One has $f * \phi = (f_0 * g_0) * f * \phi = 0$. This implies $\lambda \notin sp^A(\phi)$ and gives $sp^A(\phi) \subset sp^B(\phi)$.

The converse is proved similarly. \P

Remark 1.3. *Assume $\mathcal{A} \subset L^\infty(\mathbb{J}, X)$ is a BUC-invariant subspace (see (1.12)).*

(i) $C_0(\mathbb{J}, X) \subset \mathcal{A}$ if $\mathbb{J} = \mathbb{R}_+$ but this is not necessarily true if $\mathbb{J} = \mathbb{R}$.

(ii) If $\mathbb{J} = \mathbb{R}_+$, $\tilde{\phi} \in L^1_{loc}(\mathbb{R}, X)$, $\tilde{\phi}|_{\mathbb{R}_+} = \phi \in L^1_{loc}(\mathbb{R}_+, X) \cap \mathcal{S}'(\mathbb{R}, X)$ and $\tilde{\phi}|_{(-\infty, 0]} \in L^\infty((-\infty, 0], X)$, then $sp_{\mathcal{A}}(\tilde{\phi}) = sp_{\mathcal{A}}(\phi)$.

(iii) If $\mathcal{A} \subset BUC(\mathbb{J}, X)$ and $\phi \in L^\infty(\mathbb{R}, X)$ then definition (1.13) is equivalent to Definition 4.1.2 of [5]. In particular, $sp_{\{0\}}(\phi) = sp^B(\phi)$.

(iv) If $\mathcal{A} = C_0(\mathbb{J}, X)$ and $\phi \in L^1(\mathbb{R}, X)$, then $sp_{\mathcal{A}}(\phi) = \emptyset$.

Proof. (i) By the assumption if $\phi \in BUC(\mathbb{R}, X)$ and ϕ has compact support from $(-\infty, 0]$, then $\phi_t|_{\mathbb{R}_+} \in \mathcal{A}$ for all $t \in \mathbb{R}$. It follows $C_c(\mathbb{R}_+, X) \subset \mathcal{A}$ and so $C_0(\mathbb{R}_+, X) \subset \mathcal{A}$ (see also the proof of Theorem 2.2.4 in [5, p. 13]). A counter example for the case $\mathbb{J} = \mathbb{R}$ is $\mathcal{A} = AP(\mathbb{R}, X)$.

(ii) For $f \in \mathcal{S}(\mathbb{R})$, one has $\tilde{\phi} * f|_{\mathbb{R}_+}(t) = \int_0^\infty \phi(s)f(t-s)ds + \int_{-\infty}^0 \tilde{\phi}(s)f(t-s)ds = \int_0^\infty \phi(s)f(t-s)ds + \xi(t)$, where $\xi \in C_0(\mathbb{R}_+, X)$. This proves (ii).

(iii) We prove that a point $\omega_0 \in \mathbb{R}$ is \mathcal{A} -regular for ϕ if there is $f_0 \in L^1(\mathbb{R})$ such that $\hat{f}_0(\omega_0) \neq 0$ and $\phi * f_0|_{\mathbb{J}} \in \mathcal{A}$, since the converse is obvious. Choose $\delta > 0$ such that $\hat{f}_0(\omega) \neq 0$ for $\omega \in [\omega_0 - \delta, \omega_0 + \delta]$ and by Proposition 1.1, $g_0 \in L^1(\mathbb{R})$ such that $\hat{g}_0(\omega) \cdot \hat{f}_0(\omega) = 1$ for $\omega \in [\omega_0 - \delta, \omega_0 + \delta]$. Let $f \in \mathcal{S}(\mathbb{R})$, $\hat{f}(\omega_0) \neq 0$ and supp

$\hat{f} \subset [\omega_0 - \delta, \omega_0 + \delta]$. Using Bochner integration we have $(\phi * f_0) * g_0|_{\mathbb{J}} \in \mathcal{A}$ and so $\phi * f|_{\mathbb{J}} = \phi * (f_0 * g_0 * f)|_{\mathbb{J}} = (\phi * f_0 * g_0) * f|_{\mathbb{J}} \in \mathcal{A}$. This proves the first part and the second part follows taking $\mathcal{A} = \{0\}$.

(iv) Since $\phi \in L^1(\mathbb{R}, X)$, it follows $\phi * f \in C_0(\mathbb{R}, X)$ for all $f \in C_0(\mathbb{R})$. Since $\mathcal{S}(\mathbb{R}) \subset C_0(\mathbb{R})$ the definitions imply $sp_{C_0(\mathbb{R}, X)}(\phi) = \emptyset$ and $sp_{C_0(\mathbb{R}_+, X)}(\phi) = \emptyset$. \P

In Propositions 1.4, 1.5 ψ will denote an element of $\mathcal{S}(\mathbb{R})$ with the following properties:

(1.20) $\hat{\psi}$ has compact support, $\hat{\psi}(0) = 1$ and $\psi, \hat{\psi}$ are non-negative.

An example of such ψ is given by $\psi = \hat{\varphi}^2$, where $\varphi(t) = a e^{\frac{1}{t^2-1}}$, $|t| \leq 1$, $\varphi = 0$ elsewhere on \mathbb{R} with a some suitable constant.

Proposition 1.4. *The sequence $f_n(t) = n\psi(nt)$ is an approximate identity for the space of uniformly continuous functions $UC(\mathbb{R}, X)$, that is $\|\phi * f_n - \phi\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Given $\phi \in UC(\mathbb{R}, X)$ and $\varepsilon > 0$ there exists $k > 0$ such that $\|\phi(t+s) - \phi(t)\| \leq k|s| + \varepsilon$ for all $t, s \in \mathbb{R}$. But $\phi * f_n(t) - \phi(t) = \int_{-\infty}^{\infty} [\phi(t - \frac{s}{n}) - \phi(t)]\psi(s) ds$ which gives $\|\phi * f_n - \phi\|_{\infty} \leq (k/n) \int_{-\infty}^{\infty} |s|\psi(s) ds + \varepsilon \int_{-\infty}^{\infty} \psi(s) ds$. The result follows. \P

Proposition 1.5. *Let $\phi \in L^{\infty}(\mathbb{J}, X)$ or $\phi \in UC(\mathbb{J}, X)$. Assume $0 \notin sp_{C_0(\mathbb{J}, X)}(\phi)$. Then ϕ is bounded and ergodic with mean 0, in other words $\phi \in \mathcal{E}_0(\mathbb{J}, X)$.*

Proof. By the Definition above (1.13), $0 \notin sp_{C_0(\mathbb{J}, X)}(\phi)$ implies there is $f_0 \in \mathcal{S}(\mathbb{R})$ such that $\hat{f}_0(0) \neq 0$ and $\phi * f_0|_{\mathbb{J}} \in C_0(\mathbb{J}, X)$. Choose $\varepsilon > 0$ such that $\hat{f}_0(\omega) \neq 0$ for all $\omega \in [-2\varepsilon, 2\varepsilon]$ and $sp_{C_0(\mathbb{J}, X)}(\phi) \cap [-2\varepsilon, 2\varepsilon] = \emptyset$. By Proposition 1.1, there is $g_0 \in L^1(\mathbb{R})$ such that $\text{supp } \hat{g}_0 \subset [-2\varepsilon, 2\varepsilon]$ and $\hat{f}_0(\omega)\hat{g}_0(\omega) = 1$ for $\omega \in [-\varepsilon, \varepsilon]$. Set $f = f_0 * g_0$. If $\mathbb{J} = \mathbb{R}_+$, let $\tilde{\phi} = \phi$ on \mathbb{R}_+ and $\tilde{\phi} = \phi(0)$ elsewhere on \mathbb{R} . It follows as in the proof of Remark 1.1 (ii) that $\tilde{\phi} * f_0|_{\mathbb{J}} \in C_0(\mathbb{J}, X)$ and using Bochner integration $\tilde{\phi} * f|_{\mathbb{J}} = (\tilde{\phi} * f_0) * g_0|_{\mathbb{J}} \in C_0(\mathbb{J}, X)$. Hence $\tilde{\phi} * f|_{\mathbb{J}} \in \mathcal{E}_{u,0}(\mathbb{J}, X)$. Set $F = \tilde{\phi} - \tilde{\phi} * f$. Then $0 \notin sp_{\{0\}}(F)$. It follows F is bounded, by [8, Theorem 4.2]. Hence PF is bounded by [8, Corollary 4.4] and so $F \in \mathcal{E}_0(\mathbb{R}, X)$. This implies $\phi = [F + \tilde{\phi} * f]|_{\mathbb{J}} \in \mathcal{E}_0(\mathbb{J}, X)$. \P

Remark 1.6. *Proposition 1.5 is true for $\phi \in L_{loc}^1(\mathbb{J}, X)$ with $\Delta_h^n \phi \in BUC(\mathbb{J}, X)$ for all $h > 0$ and some $n \in \mathbb{N}$. Here $\Delta_h^n \phi = \Delta_h(\Delta_h^{n-1} \phi)$ for $n > 1$. However, it is not valid for arbitrary $\phi \in L_{loc}^1(\mathbb{J}, X) \cap \mathcal{S}'(\mathbb{R}, X)$ as the following example shows. The function $\phi(t) = te^{it}$ has Beurling spectrum $sp^B(\phi) = \{1\}$ and a direct calculation shows that ϕ is not ergodic.*

§2. LAPLACE AND WEAK LAPLACE SPECTRA

In this section we establish some tools which enable us to calculate Laplace and weak Laplace spectra and relate them to the reduced Beurling spectrum relative to a class \mathcal{A} satisfying (1.12). We prove new properties (Theorems 2.3(i), 2.4(i)) of the weak Laplace spectrum which enable us to give simple proofs of several tauberian results of Ingham [20], [22] and their generalizations by Chill and others (see [4, 4.10, p. 332] and references therein).

Proposition 2.1. *If $\Phi \in L_{loc}^1(\mathbb{J}, X) \cap \mathcal{S}'(\mathbb{R}, X)$ or $\Phi \in L_s^\infty(\mathbb{J}, L(X))$, then*

- (i) $sp^{\mathcal{L}}(\Phi) = sp^{\mathcal{L}}(\Phi_a)$ for each $a \in \mathbb{J}$.
- (ii) $sp^{\mathcal{L}}(\Phi) = \cup_{h>0} sp^{\mathcal{L}}(M_h \Phi)$.
- (iii) $sp^{\mathcal{L}}(\gamma_\omega \Phi) = \omega + sp^{\mathcal{L}}(\Phi)$.
- (iv) *The statements (i), (ii), (iii) hold true for $sp^{w\mathcal{L}}$ and, when $\mathbb{J} = \mathbb{R}$, for $sp^{\mathcal{C}}$.*

Proof. (i) A simple calculation shows for $\lambda \in \mathbb{C}^\pm$ and $\mathbb{J} = \mathbb{R}$

$$(2.1) \quad \mathcal{L}^\pm \Phi_a(\lambda) = e^{\lambda a} \mathcal{L}^\pm \Phi(\lambda) - e^{\lambda a} \int_0^a e^{-\lambda t} \Phi(t) dt.$$

It follows $\mathcal{L}^+ \Phi$ (respectively $\mathcal{C}\Phi$) is holomorphic at $i\omega$ if and only if $\mathcal{L}^+ \Phi_a$ (respectively $\mathcal{C}\Phi_a$) is holomorphic at $i\omega$. This proves (i) for $sp^{\mathcal{L}}$ and $sp^{\mathcal{C}}$.

(ii) Another calculation shows for $\lambda \in \mathbb{C}^\pm$ and $\mathbb{J} = \mathbb{R}$

$$(2.2) \quad \mathcal{L}^\pm(M_h \Phi)(\lambda) = g(\lambda h) \mathcal{L}^\pm \Phi(\lambda) - (1/h) \int_0^h (e^{\lambda v} \int_0^u e^{-\lambda t} \Phi(t+v) dt) dv,$$

where g is the entire function given by $g(\lambda) = \frac{e^\lambda - 1}{\lambda}$ for $\lambda \neq 0$. Let $i\omega \in i\mathbb{R}$ be a regular point for $\mathcal{L}^+ \Phi$ and let $\overline{\mathcal{L}^+} \Phi : V \rightarrow X$ be a holomorphic extension of $\mathcal{L}^+ \Phi$ to a neighbourhood $V \subset \mathbb{C}$ of $i\omega$. Then $\overline{\mathcal{L}^+}(M_h \Phi)(\lambda) = g(\lambda h) \overline{\mathcal{L}^+} \Phi(\lambda) - (1/h) \int_0^h (e^{\lambda v} \int_0^u e^{-\lambda t} \Phi(t+v) dt) dv$, $\lambda \in V$ is a holomorphic extension of $\mathcal{L}^+(M_h \Phi)$. So $i\omega$ is a regular point for $\mathcal{L}^+(M_h \Phi)$. If $\omega \in \mathbb{R}$ there is $h_0 > 0$ such that $g(i\omega h_0) \neq 0$. Similarly as above, if $i\omega \in i\mathbb{R}$ is a regular point for $\mathcal{L}^+(M_{h_0} \Phi)$,

then $i\omega$ is a regular point for $\mathcal{L}^+\Phi$. This proves the first part of (ii). The second part follows similarly noting that (2.2) implies $\mathcal{C}(M_h\Phi)(\lambda) = g(\lambda h)\mathcal{C}\Phi(\lambda) - (1/h)\int_0^h(e^{\lambda v}\int_0^u e^{-\lambda t}\Phi(t+v)dt)dv$. This proves (ii) for $sp^{\mathcal{L}}$ and $sp^{\mathcal{C}}$.

(iii) This follows easily from the definitions noting that $\mathcal{L}(\gamma_\omega\Phi)(\lambda) = \mathcal{L}\Phi(\lambda - i\omega)$ and $\mathcal{C}(\gamma_\omega\Phi)(\lambda) = \mathcal{C}\Phi(\lambda - i\omega)$. This proves (iii) for $sp^{\mathcal{L}}$, $sp^{\mathcal{C}}$ and $sp^{w\mathcal{L}}$. ¶

(iv) The proofs of (i) and (ii) for $sp^{w\mathcal{L}}$ follow similarly as in the case $sp^{\mathcal{L}}$ using [28, Theorem 6.18, p.146] as in the proof of Proposition 2.4(i) below.

Example 2.2. Let $\phi(t) = e^{it^2}$, $t \in \mathbb{R}$. Then $sp^{\mathcal{L}}(\phi) = \emptyset$ and $sp^{\mathcal{C}}(\phi) = \mathbb{R}$. Moreover, $M_h\phi \in C_0(\mathbb{R}, \mathbb{C})$ and $sp^{\mathcal{L}}(M_h\phi) = \emptyset$ for $h > 0$.

Proof. By Proposition 1.1 (i), (iii), it is readily verified that $sp^{\mathcal{L}}(\phi_a) = sp^{\mathcal{L}}(\phi) = 2a + sp^{\mathcal{L}}(\phi)$ for each $a \in \mathbb{R}$. This implies that either $sp^{\mathcal{L}}(\phi) = \emptyset$ or $sp^{\mathcal{L}}(\phi) = \mathbb{R}$. We claim that $sp^{\mathcal{L}}(\phi) = \emptyset$. Indeed, let $y(\lambda) = \mathcal{L}\phi(\lambda)$ for $\lambda \in \mathbb{C}_+$. Then $y'(\lambda) + (\lambda/2i)y(\lambda) = 1/2i$. Solving this equation, one gets $\psi(\lambda, a) = e^{-(\lambda^2/4i)}(a + (1/2i)Pe^{(\lambda^2/4i)})$ is a general solution, where $a \in \mathbb{C}$. One has $\mathcal{L}\phi(0) = ((1 + i)\pi^{1/2})/2^{3/2} = a_0$ and so $\mathcal{L}\phi(\lambda) = \psi(\lambda, a_0)$ is a particular solution. Since $e^{(\lambda^2/4i)}$ is an entire function, $\psi(\lambda, a_0)$ is an entire extension of $\mathcal{L}\phi$, implying $sp^{\mathcal{L}}(\phi) = \emptyset$. A similar argument shows that either $sp^{\mathcal{C}}(\phi) = \emptyset$ or $sp^{\mathcal{C}}(\phi) = \mathbb{R}$. Since $sp^{\mathcal{B}}(\phi) = sp^{\mathcal{C}}(\phi) \neq \emptyset$, one gets $sp^{\mathcal{C}}(\phi) = \mathbb{R}$. Since $\int_0^\infty e^{it^2} dt$ is an improper Riemann integral, it follows $P\phi(T) = \int_0^T e^{it^2} dt \rightarrow a_0$ as $T \rightarrow \infty$. Since $M_h\phi(t) = P\phi(t+h) - P\phi(t)$, $M_h\phi \in C_0(\mathbb{R}, \mathbb{C})$ for $h > 0$. Finally, by Proposition 2.1(ii) and $sp^{\mathcal{L}}(\phi) = \emptyset$, one concludes $sp^{\mathcal{L}}(M_h\phi) = \emptyset$ for $h > 0$. ¶

Theorem 2.3 (Ingham). Let $\phi \in L^1_{loc}(\mathbb{R}_+, X) \cap \mathcal{S}'(\mathbb{R}, X)$ and $sp^{w\mathcal{L}}(\phi) = \emptyset$.

- (i) $\phi * g \in C_0(\mathbb{R}, X)$ for all $g \in \mathcal{S}(\mathbb{R})$ with $\hat{g} \in \mathcal{D}(\mathbb{R})$.
- (ii) If $\phi \in L^\infty(\mathbb{R}_+, X)$, then $\phi * g \in C_0(\mathbb{R}, X)$ for all $g \in L^1(\mathbb{R})$.
- (iii) If $\phi \in BUC(\mathbb{R}_+, X)$ or more generally ϕ is slowly oscillating, then $\phi \in C_0(\mathbb{R}_+, X)$.

(iv) If $sp^{\mathcal{L}}(\phi) = \emptyset$, then $(P\gamma_{-\omega}\phi - \overline{\mathcal{L}}\phi(i\omega)) \in C_0(\mathbb{R}_+, X)$ for all $\omega \in \mathbb{R}$.

Proof. (i) As $\phi \in \mathcal{S}'(\mathbb{R}, X)$, the Fourier transform $\hat{\phi} \in \mathcal{S}'(\mathbb{R}, X)$ is given by

$$(2.3) \quad \langle \hat{\phi}, f \rangle = \langle \phi, \hat{f} \rangle = \int_0^\infty \phi(s)\hat{f}(s)ds, \quad f \in \mathcal{S}(\mathbb{R}).$$

The condition $sp^{w\mathcal{L}}(\phi) = \emptyset$ means that $\widehat{\phi}|_{\mathcal{D}(\mathbb{R})} = h$, where h is locally integrable on \mathbb{R} (see for example [4, Lemma 4.9.3]) and so

$$(2.4) \quad \int_0^\infty \phi(s)\widehat{f}(s)ds = \int_{-\infty}^\infty h(\eta)f(\eta)d\eta, \quad f \in \mathcal{D}(\mathbb{R}).$$

Take $g \in \mathcal{S}(\mathbb{R})$ with $\widehat{g} \in \mathcal{D}(\mathbb{R})$. Using (2.3) and (2.4), one gets

$$(2.5) \quad \phi * g(t) = \int_0^\infty \phi(s)g(t-s)ds = (1/2\pi) \int_{-\infty}^\infty h(\eta)e^{it\eta}\widehat{g}(\eta)d\eta.$$

Now, $h\widehat{g} \in L^1(\mathbb{R}, X)$ and so by the Riemann-Lebesgue lemma (see [4, p. 45] or [21, p. 123]), $\phi * g \in C_0(\mathbb{R}, X)$. This proves (i).

(ii) Given $g \in L^1(\mathbb{R})$ choose $(g_n) \subset \mathcal{S}(\mathbb{R})$ with $(\widehat{g_n}) \subset \mathcal{D}(\mathbb{R})$ and $\|g_n - g\|_{L^1} \rightarrow 0$ as $n \rightarrow \infty$. Since $\phi \in L^\infty(\mathbb{R}_+, X)$, we conclude $\|\phi * g_n - \phi * g\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ and so by (i), $\phi * g \in C_0(\mathbb{R}, X)$. This proves (ii).

(iii) Consider first the case $\phi \in BUC(\mathbb{R}_+, X)$ and let $\omega \in \mathbb{R}$, $\varepsilon > 0$. Take $g \in \mathcal{S}(\mathbb{R})$ with $\text{supp } \widehat{g} \subset [\omega - \varepsilon, \omega + \varepsilon]$ and $\widehat{g}(\omega) \neq 0$. By (i), $\phi * g \in C_0(\mathbb{R}, X)$. By Remark 1.1 (ii), we conclude $\omega \notin sp_{C_0(\mathbb{R}_+, X)}(\tilde{\phi})$, where $\tilde{\phi} \in BUC(\mathbb{R}, X)$ is any extension of ϕ . This implies $sp_{C_0(\mathbb{R}_+, X)}(\tilde{\phi}) = \emptyset$ and so $\phi = \tilde{\phi}|_{\mathbb{R}_+} \in C_0(\mathbb{R}_+, X)$ by [5, Theorem 4.2.1]. For the general case, it follows $sp_{C_0(\mathbb{R}_+, X)}(M_h\phi) = \emptyset$ for all $h > 0$ and hence $M_h\phi \in C_0(\mathbb{R}_+, X)$ for all $h > 0$. This completes the proof of (iii), by [4, Theorem 3.2.3, p. 250].

(iv) Replacing ϕ by $\gamma_{-\omega}\phi$, we may assume $\omega = 0$. Set $\Psi(t) = P\phi(t) - \overline{\mathcal{L}}\phi(0)$ for $t \in \mathbb{R}_+$ and $\Psi = -\overline{\mathcal{L}}\phi(0)$ elsewhere on \mathbb{R} . Simple calculation shows $\mathcal{L}\Psi(\lambda) = (\mathcal{L}\phi(\lambda) - \overline{\mathcal{L}}\phi(0))/\lambda$ for $\lambda \in \mathbb{C}_+$. Since $\overline{\mathcal{L}}\phi$ is holomorphic on $\overline{\mathbb{C}_+}$, it follows that $\mathcal{L}\Psi$ has a holomorphic extension to an open neighbourhood of $\overline{\mathbb{C}_+}$. This implies $sp^\mathcal{L}(\Psi) = \emptyset$. So, by (1.11) and part (i), one gets $\Psi * g|_{\mathbb{R}_+} \in C_0(\mathbb{R}_+, X)$ for all $g \in \mathcal{S}(\mathbb{R})$ with $\widehat{g} \in \mathcal{D}(\mathbb{R})$. Consider the sequence $f_n(t)$ of Proposition 1.3. Since $\Psi \in UC(\mathbb{R}, X)$ and $\Psi * f_n|_{\mathbb{R}_+} \in C_0(\mathbb{R}_+, X)$ for each $n \in \mathbb{N}$, one concludes $\Psi|_{\mathbb{R}_+} \in C_0(\mathbb{R}_+, X)$ by Proposition 1.3. This finishes the proof. \P

Theorem 2.4. *Let $\phi \in L^1_{loc}(\mathbb{R}_+, X) \cap \mathcal{S}'(\mathbb{R}, X)$. Then*

$$(i) \quad sp^{w\mathcal{L}}(\phi * f) \subset sp^{w\mathcal{L}}(\phi) \cap \text{supp } \widehat{f} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}).$$

$$(ii) \quad sp_{C_0(\mathbb{R}_+, X)}(\phi) \subset sp^{w\mathcal{L}}(\phi).$$

$$(iii) \quad \text{If } \phi \in L^\infty(\mathbb{R}_+, X) \text{ and } \omega_0 \notin sp^{w\mathcal{L}}(\phi), \text{ then } \gamma_{-\omega_0}\phi \in \mathcal{E}_0(\mathbb{R}_+, X).$$

$$(iv) \quad \text{If } \phi \in L^\infty(\mathbb{R}_+, X) \text{ and } \omega_0 \notin sp^\mathcal{L}(\phi), \text{ then } (P\gamma_{-\omega_0}\phi - \overline{\mathcal{L}}\phi(i\omega_0)) \in C_0(\mathbb{R}_+, X).$$

Proof. (i) Direct calculations show that for $\lambda \in \mathbb{C}$

$$\mathcal{L}\phi * f(\lambda) = \mathcal{L}\phi(\lambda) \cdot \mathcal{L}f(\lambda) + \zeta(\lambda) \text{ with } \zeta(\lambda) = \int_0^\infty \mathcal{L}\phi_s(\lambda) f(-s) ds.$$

Since $f \in \mathcal{S}(\mathbb{R})$, $\lim_{a \searrow 0} \mathcal{L}f(a + i\eta) = \int_0^\infty e^{-i\eta t} f(t) dt := g(\eta)$ for each $\eta \in \mathbb{R}$. Since $g \in C^\infty(\mathbb{R})$ we conclude $sp^{w\mathcal{L}}(f) = \emptyset$. Now, assume $i\omega$ is a weak regular point for $\mathcal{L}\phi$. So there exists $\varepsilon > 0$ and $h \in L^1(\omega - \varepsilon, \omega + \varepsilon)$ satisfying $\lim_{a \searrow 0} \int_{-\infty}^\infty \mathcal{L}\phi(a + i\eta) \varphi(\eta) d\eta = \int_{i\omega - \varepsilon}^{i\omega + \varepsilon} h(s) \varphi(\eta) d\eta$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp } \varphi \subset [\omega - \varepsilon, \omega + \varepsilon]$. Then by [28, Theorem 6.18, p. 146], $\lim_{a \searrow 0} \int_{-\infty}^\infty \mathcal{L}\phi(a + i\eta) \mathcal{L}f(a + i\eta) \varphi(\eta) d\eta = \int_{\omega - \varepsilon}^{\omega + \varepsilon} h(\eta) g(\eta) \varphi(\eta) d\eta$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp } \varphi \subset [\omega - \varepsilon, \omega + \varepsilon]$. It follows $i\omega$ is a weak regular point for $\mathcal{L}\phi \cdot \mathcal{L}f$. Since $\mathcal{L}\phi_s(a + i\eta) = e^{(a+i\eta)s} [\mathcal{L}\phi(a + i\eta) - \int_0^s e^{-(a+i\eta)t} \phi(t) dt]$, the same argument shows ω is a weak regular point for ζ . It follows ω is a weak regular point for $\mathcal{L}(\phi * f)$ showing $sp^{w\mathcal{L}}(\phi * f) \subset sp^{w\mathcal{L}}(\phi)$. By (1.11), (1.14), one gets $sp^{w\mathcal{L}}(\phi * f) \subset sp^{\mathcal{C}}(\phi * f) \subset \text{supp } \hat{f}$. This finishes the proof of (i).

(ii) Let $\omega \notin sp^{w\mathcal{L}}(\phi)$. Choose $\varepsilon > 0$ and $f \in \mathcal{S}(\mathbb{R})$ such that $sp^{w\mathcal{L}}(\phi) \cap [\omega - \varepsilon, \omega + \varepsilon] = \emptyset$, $\hat{f}(\omega) = 1$ and $\text{supp } \hat{f} \subset [\omega - \varepsilon, \omega + \varepsilon]$. By part (i), it follows $sp^{w\mathcal{L}}(\phi * f) = \emptyset$ and so by Theorem 2.3 (ii), $\phi * f|_{\mathbb{R}_+} \in C_0(\mathbb{R}_+, X)$. This implies $\omega \notin sp_{C_0(\mathbb{R}_+, X)}(\phi)$ and proves (ii).

(iii) Replacing ϕ by $\gamma_{-\omega_0} \phi$, we may assume $\omega_0 = 0$. By part (ii), one concludes $0 \notin sp_{C_0(\mathbb{R}_+, X)}(\phi)$. The statement follows by Proposition 1.4.

(iv) As in the proof of Theorem 2.3 (iv), we may assume $\omega_0 = 0$ and conclude $0 \notin sp^{\mathcal{L}}(P\phi - \overline{\mathcal{L}}\phi(0)) = sp^{\mathcal{L}}(\phi)$. Choose $\varepsilon > 0$ and $f \in \mathcal{S}(\mathbb{R})$ such that $sp^{\mathcal{L}}(\phi) \cap [-\varepsilon, \varepsilon] = \emptyset$, $\text{supp } \hat{f} \subset [-\varepsilon, \varepsilon]$ and $\hat{f}(\omega) = 1$ if $|\omega| \leq \varepsilon/2$. One has $(P\phi * f - \overline{\mathcal{L}}\phi(0)) = (P\phi - \overline{\mathcal{L}}\phi(0)) * f$ and so by Part (i), it follows $sp^{w\mathcal{L}}(P\phi * f - \overline{\mathcal{L}}\phi(0)) = \emptyset$. Choose $g \in \mathcal{S}(\mathbb{R})$ with $\hat{g} \in \mathcal{D}(\mathbb{R})$ and $\hat{g}(\omega) = 1$ if $|\omega| \leq \varepsilon$. Then $(P\phi * f - \overline{\mathcal{L}}\phi(0)) = (P\phi * f - \overline{\mathcal{L}}\phi(0)) * g \in C_0(\mathbb{R}_+, X)$, by Theorem 2.3(i). It is easily verified that $0 \notin sp^{\mathcal{C}}(\phi - \phi * f)$ and hence $P(\phi - \phi * f) \in BUC(\mathbb{R}, X)$, by [8, Corollary 4.4]. It follows $P\phi = (P(\phi - \phi * f) + P\phi * f)|_{\mathbb{R}_+} \in BUC(\mathbb{R}_+, X)$. \P

We conclude this section giving a short proof for [4, Theorem 4.9.7].

Theorem 2.5. *Let $\phi \in BUC(\mathbb{R}_+, X)$ or more generally $\phi \in L^\infty(\mathbb{R}_+, X)$ with ϕ slowly oscillating. If $sp^{w\mathcal{L}}(\phi)$ is countable and $\gamma_{-\omega}\phi \in \mathcal{E}(\mathbb{R}_+, X)$ for all $\omega \in$*

$sp^{w\mathcal{L}}(\phi)$, then $\phi \in AAP(\mathbb{R}_+, X)$.

Proof. Consider first the case $\phi \in BUC(\mathbb{R}_+, X)$. Let $\tilde{\phi} \in BUC(\mathbb{R}, X)$ be an extension of ϕ . Using $C_0(\mathbb{R}_+, X) \subset AAP(\mathbb{R}_+, X)$, (1.11) and Theorem 2.4 (ii), one gets $sp_{AAP(\mathbb{R}_+, X)}(\tilde{\phi}) \subset sp_{C_0(\mathbb{R}_+, X)}(\tilde{\phi}) \subset sp^{w\mathcal{L}}(\tilde{\phi}) = sp^{w\mathcal{L}}(\phi)$. It follows, $sp_{AAP(\mathbb{R}_+, X)}(\tilde{\phi})$ is countable. This and the assumptions imply $\phi = \tilde{\phi}|_{\mathbb{R}_+} \in AAP(\mathbb{R}_+, X)$, by [5, Theorem 4.2.6]. The case ϕ is slowly oscillating can be proved similarly to Theorem 2.3(ii) noting that direct calculation shows that if $\gamma_{-\omega}\phi \in \mathcal{E}(\mathbb{R}_+, X)$, then $\gamma_{-\omega}M_h\phi \in \mathcal{E}(\mathbb{R}_+, X)$ for all $h > 0$. \P

§3. BOUNDED C_0 -SEMIGROUPS (GROUPS)

For a bounded C_0 -group $T(t) \in L(X)$, $t \in \mathbb{R}$, we make the following definitions recalling that $T(\cdot)$ is strongly measurable but not necessarily measurable:

$$(3.1) \quad sp^B(T(\cdot)) := \{\lambda \in \mathbb{R} : \text{there is } y \in X \text{ such that if } f \in L^1(\mathbb{R}) \text{ and } \hat{f}(\lambda) = 1 \text{ then } f * T(\cdot)y \neq 0\}.$$

The Arveson spectrum of $T(\cdot)$ is defined ([2, p. 365], [14, Definition 4]) by

$$(3.2) \quad sp^A(T(\cdot)) = \{\lambda \in \mathbb{R} : \text{for each } \varepsilon > 0 \text{ there is } y \in X \text{ and } f \in L^1(\mathbb{R}) \text{ with } \text{supp } \hat{f} \subset]\lambda - \varepsilon, \lambda + \varepsilon[\text{ and } f * T(\cdot)y \neq 0\}.$$

Using (1.17), it is easily verified that

$$(3.3) \quad sp^B(T(\cdot)) = \cup_{x \in X} sp^B(T(\cdot)x).$$

Proposition 3.1. *Let $T(t) \in L(X)$, $t \in \mathbb{R}_+$ be a bounded C_0 -semigroup with generator A . Then*

$$(3.4) \quad i sp^{\mathcal{L}}(T(\cdot)) = \sigma(A) \cap i\mathbb{R} = i \cup_{x \in X} sp^{\mathcal{L}}(T(\cdot)x).$$

Proof. By [25, (3.5), (3.7), p. 9], one has $(\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t}T(t)x dt$ for $\lambda \in \mathbb{C}_+$, $x \in X$. It follows by (1.1) and (1.2) that $(\lambda - A)^{-1} = \mathcal{L}T(\cdot)$. Since $R(\lambda) := (\lambda - A)^{-1}$ is holomorphic on $\rho(A) = \mathbb{C} \setminus \sigma(A)$ (see [25, (5.21), p. 20]), it follows that if $i\omega \in \rho(A) \cap i\mathbb{R}$ then $\omega \notin sp^{\mathcal{L}}(T(\cdot))$ proving $i sp^{\mathcal{L}}(T(\cdot)) \subset \sigma(A) \cap i\mathbb{R}$. The definitions imply $\cup_{x \in X} sp^{\mathcal{L}}(T(\cdot)x) \subset sp^{\mathcal{L}}(T(\cdot))$. So, it remains to show $\sigma(A) \cap i\mathbb{R} \subset i \cup_{x \in X} sp^{\mathcal{L}}(T(\cdot)x)$. Assume $\omega \in \mathbb{R} \setminus \cup_{x \in X} sp^{\mathcal{L}}(T(\cdot)x)$. Then for each $x \in X$, there is an open disk $V_x \subset \mathbb{C}$ with center $i\omega$ and a holomorphic function $F_x : V_x \rightarrow X$ such that $F_x(\lambda) = \mathcal{L}T(\cdot)x(\lambda) = R(\lambda)x$ for $\lambda \in V_x \cap \mathbb{C}_+$. It follows

$(\lambda I - A)F_x(\lambda) = x$, for $\lambda \in V_x \cap \mathbb{C}_+$, $x \in X$ and $\lambda F_x(\lambda) - F_{Ax}(\lambda) = x$, for $\lambda \in V_x \cap V_{Ax} \cap \mathbb{C}_+$, $x \in D(A)$. Since F_x is continuous at $i\omega$ for each $x \in X$ and A is a closed operator, the identities remain valid for $\lambda = i\omega$. The identity $(i\omega I - A)F_x(i\omega) = x$ implies $i\omega I - A$ is onto. Now, assume $x_0 \in D(A)$ with $(i\omega I - A)x_0 = 0$. Then $i\omega F_{x_0}(i\omega) - F_{Ax_0}(i\omega) = x_0 = F_0(i\omega) = 0$. This gives $x_0 = 0$ and proves $i\omega I - A$ is one to one. By the closed graph theorem $i\omega \in \rho(A)$. This completes the proof. \blacksquare

We note that if $g \in L^1(\mathbb{R})$ with $\hat{g}(\lambda) = 1$, $\text{supp } \hat{g} \subset [\lambda - \delta, \lambda + \delta]$, $T(t) \in L(X)$, $t \in \mathbb{R}$ is a bounded C_0 -group, $x \in X$ and $y = \int_{-\infty}^{\infty} g(s)T(-s)x ds$, then

$$(3.5) \quad T(\cdot)y = g * T(\cdot)x \text{ and } sp^B(T(\cdot)y) \subset [\lambda - \delta, \lambda + \delta].$$

For $y \in X$ set $X_y := \overline{\text{span}}\{z \in X : z = T(t)y, t \in \mathbb{R}\}$, $T_y(t)z = T(t)z$ for all $z \in X_y$ and A_y the generator of $T_y(\cdot)$.

Proposition 3.2. *Let $T(t) \in L(X)$, $t \in \mathbb{R}$ be a bounded C_0 -group and $x \in X$.*

$$(3.6) \quad isp^B(T(\cdot)x) = isp^C(T(\cdot)x) = isp^L(T(\cdot)x) = \sigma(A_x),$$

$$(3.7) \quad isp^A(T(\cdot)) = isp^B(T(\cdot)) = isp^C(T(\cdot)) = isp^L(T(\cdot)) = \sigma(A).$$

Furthermore each of these sets is closed.

Proof. By the first equality of (1.19) for $\phi = T(\cdot)x$, one gets $sp^B(T(\cdot))x$ is closed for each $x \in X$. We prove $sp^B(T(\cdot))$ is closed. Let $(\lambda_n) \subset sp^B(T(\cdot))$ and $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. We restrict ourself to the case $\lambda_n > \lambda_{n+1}$ for $n \in \mathbb{N}$. Choose $x_n \in X$ such that $\|x_n\| = 1$ and $f * T(\cdot)x_n \neq 0$ for each $f \in L^1(\mathbb{R})$ with $\hat{f}(\lambda_n) = 1$. By (3.5), one can replace (x_n) by (y_n) with $sp^B(T(\cdot)y_n) \subset I_n = [\lambda_n - \delta_n, \lambda_n + \delta_n]$, where $0 < \delta_n < (\lambda_n - \lambda_{n+1})/2$ and $\delta_n > \delta_{n+1}$ for $n \in \mathbb{N}$. Set $y = \sum_{k=1}^{\infty} y_k/2^k$. Because the intervals I_n are compact and disjoint, it follows that choosing $h_n \in L^1(\mathbb{R})$ with $\hat{h}_n = 1$ on I_n and $\hat{h}_n = 0$ on a neighbourhood of $\cup_{k \neq n} I_k$ one has $h_n * T(\cdot)y = (T(\cdot)y_n)/2^n$. Hence $sp^B(T(\cdot)y_n) \subset sp^B(T(\cdot)y)$ for each $n \in \mathbb{N}$. Let $f \in L^1(\mathbb{R})$ and $\hat{f}(\lambda) = 1$. Since $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$ and \hat{f} is continuous, there is $n(f)$ such that $\hat{f}(\lambda_n) \neq 0$ for $n \geq n(f)$. It follows $f * T(\cdot)y_n \neq 0$ for $n \geq n(f)$. This implies $f * T(\cdot)y \neq 0$ and proves $\lambda \in sp^B(T(\cdot))$ and so $sp^B(T(\cdot))$ is closed.

By (3.3) and (1.19) we get

$$(3.8) \quad sp^B(T(\cdot)) = \cup_{x \in X} sp^B(T(\cdot)x) = \cup_{x \in X} sp^A(T(\cdot)x).$$

We claim $sp^A(T(\cdot)) = \cup_{x \in X} sp^A(T(\cdot)x)$. Indeed, the definition gives $sp^A(T(\cdot)x) \subset sp^A(T(\cdot))$ for each $x \in X$ and so $\cup_{x \in X} sp^A(T(\cdot)x) \subset sp^A(T(\cdot))$. As $sp^B(T(\cdot)) = \cup_{x \in X} sp^A(T(\cdot)x)$ is closed, for each $\lambda \notin \cup_{x \in X} sp^A(T(\cdot)x)$ there is $\delta > 0$ such that $[\lambda - \delta, \lambda + \delta] \cap (\cup_{x \in X} sp^A(T(\cdot)x)) = \emptyset$. Let $f \in L^1(\mathbb{R})$ and $\text{supp } \hat{f} \subset [\lambda - \delta, \lambda + \delta]$. Then $sp^B(f*T(\cdot)x) = \emptyset$ so $f*T(\cdot)x = 0$ for each $x \in X$. This implies $\lambda \notin sp^A(T(\cdot))$, $sp^A(T(\cdot)) = \cup_{x \in X} sp^A(T(\cdot)x)$ and proves the first identity of (3.7).

As in the proof of Proposition 3.1, $T(\cdot) \in L_s^\infty(\mathbb{R}, L(X))$ and $CT(\cdot)(\lambda) = (\lambda - A)^{-1}$ for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. Moreover, if $i\omega$ is a regular point for $\mathcal{L}T(\cdot)$ then by Proposition 3.1 $i\omega \in \rho(A)$. So, $i\omega$ is a regular point for $(\lambda - A)^{-1}$. It follows $i\omega$ is a singular point for $CT(\cdot)$ if and only if $i\omega$ is a singular point for $\mathcal{L}T(\cdot)$. This and Proposition 3.1 imply $sp^C(T(\cdot)) = sp^L(T(\cdot)) = \sigma(A) \cap i\mathbb{R}$. Noting that $sp^L(T(\cdot)z) \subset sp^L(T(\cdot)x)$ for each $z \in X_x$ and using Proposition 3.1 one gets $\sigma(A_x) \cap i\mathbb{R} = isp^L(T_x(\cdot)) = isp^L(T(\cdot)x)$. Since $\cup_{y \in X} sp^C(T(\cdot)y) \subset sp^C(T(\cdot)) = sp^L(T(\cdot)) = \cup_{y \in X} sp^L(T(\cdot)y) \subset \cup_{y \in X} sp^C(T(\cdot)y)$, one gets $sp^C(T(\cdot)) = \cup_{y \in X} sp^C(T(\cdot)y)$. But $sp^C(T(\cdot)z) \subset sp^C(T(\cdot)x)$ for each $z \in X_x$. So, by the above, we conclude $sp^C(T(\cdot)x) = sp^C(T_x(\cdot)) = sp^L(T_x(\cdot)) = sp^L(T(\cdot)x)$ for each $x \in X$. By (1.19), Proposition 1.2 and [4, Theorem 4.8.4] $sp^B(T(\cdot)x) = sp^C(T(\cdot)x)$ for each $x \in X$. Since $T(\cdot)$ is bounded, one gets $\sigma(A) \subset i\mathbb{R}$ and $\sigma(A_x) \subset i\mathbb{R}$ for each $x \in X$. So, (3.6) and (3.7) follow by the above, (3.4) and (3.8). ◀

Remark 3.3. (i) Definition (1.16) of $sp^A(T(\cdot)x)$ is easily seen to be equivalent to the definition of the Arveson spectrum of x with respect to $T(\cdot)$ in [2, p. 365], [3], [14, Definition 4].

(ii) In [14] a proof that $sp^A(T(\cdot)) = \overline{\cup_{x \in X} sp^A(T(\cdot)x)}$ is outlined. So, the result $sp^A(T(\cdot)) = \cup_{x \in X} sp^A(T(\cdot)x)$ seems new.

(iii) The Arveson spectrum defined in [17, p. 285], [18] is easily seen to be $i\{\lambda : \hat{f}(\lambda) = 0 \text{ for all } f \cap_{x \in X} I(T(\cdot)x)\} = \sigma(A) = isp^A(T(\cdot))$, by [17, p. 285] or [18] and Propositions 3.1, 3.2. This gives a proof of the remark above Theorem 5 in [14].

§4. UNIFORM LAPLACE AND CARLEMAN SPECTRA

In this section we recall the definition of uniform spectrum $sp^{\mathcal{L}^u}(\phi)$ for functions

from $BUC(\mathbb{J}, X)$ and extend it to functions from $L^\infty(\mathbb{J}, X)$. For $\phi \in BUC(\mathbb{J}, X)$, the Carleman spectrum coincides with Arveson spectrum of the generator A_ϕ of the group of translations $S(\cdot)$ restricted to the subspace $L(\phi) = \overline{\text{span}}\{\phi_t : t \in \mathbb{R}\}$. In the case $\phi \in BC(\mathbb{R}, X)$ it is proved in [15, Proposition 2.3 (iii)] that $sp^\mathcal{L}(\phi) \subset sp^{\mathcal{L}_u}(\phi)$ and the inclusion may be strict. Here, we prove that $sp^{\mathcal{L}_u}(\phi) = sp^\mathcal{L}(\phi)$ for $\phi \in L^\infty(\mathbb{J}, X)$ (Theorem 4.2).

This result is new and our proof seems new even for the case $\phi \in BUC(\mathbb{J}, X)$.

Using (1.1) for $\phi \in L^\infty(\mathbb{J}, X)$ and (1.3) when $\mathbb{J} = \mathbb{R}$, define

$$(4.1) \quad \mathcal{L}_u\phi(\lambda)(s) = \mathcal{L}\phi_s(\lambda) \quad \text{for } s \in \mathbb{J}, \lambda \in \mathbb{C}_+;$$

$$(4.2) \quad \mathcal{C}_u\phi(\lambda)(s) = \mathcal{C}\phi_s(\lambda) \quad \text{for } s \in \mathbb{R}, \lambda \in \mathbb{C} \setminus i\mathbb{R}.$$

Lemma 4.1. (i) For $\mathcal{L}_u\phi$, one has $\mathcal{L}_u\phi(\lambda) \in BUC(\mathbb{J}, X)$ and $\mathcal{L}_u\phi(\cdot)$ is holomorphic on \mathbb{C}_+ . If $\phi \in BUC(\mathbb{J}, X)$, then $\mathcal{L}_u\phi = \mathcal{L}S^\mathbb{J}(\cdot)\phi$.

(ii) For $\mathcal{C}_u\phi$, one has $\mathcal{C}_u\phi(\lambda) \in BUC(\mathbb{R}, X)$ and $\mathcal{C}_u\phi(\cdot)$ is holomorphic on $\mathbb{C} \setminus i\mathbb{R}$. If $\phi \in BUC(\mathbb{R}, X)$, then $\mathcal{C}_u\phi = \mathcal{C}S^\mathbb{R}(\cdot)\phi$.

(iii) If $\mathbb{J} = \mathbb{R}^+$, $\omega_0 \notin sp^\mathcal{L}(\phi)$ (or $\mathbb{J} = \mathbb{R}$, $\omega_0 \notin sp^\mathcal{C}(\phi)$) and $\mathcal{L}\phi : V \rightarrow X$ (respectively $\mathcal{C}\phi : V \rightarrow X$) is a holomorphic extension of $\mathcal{L}\phi$ (respectively $\mathcal{C}\phi$) to an open neighbourhood $V \subset \mathbb{C}$ of $i\omega_0$, then

$$(4.3) \quad \overline{\mathcal{L}}_u\phi(\lambda)(s) = e^{\lambda s}(\overline{\mathcal{L}}\phi(\lambda) - \int_0^s e^{-\lambda t}\phi(t) dt),$$

$$\overline{\mathcal{C}}_u\phi(\lambda)(s) = e^{\lambda s}(\overline{\mathcal{C}}\phi(\lambda) - \int_0^s e^{-\lambda t}\phi(t) dt)$$

are extensions of $\mathcal{L}_u\phi$ and $\mathcal{C}_u\phi$ respectively; moreover, $\overline{\mathcal{L}}_u\phi(\lambda) \in BUC(\mathbb{R}_+, X)$ and $\overline{\mathcal{C}}_u\phi(\lambda) \in BUC(\mathbb{R}, X)$ for each $\lambda \in V$.

Proof. (i),(ii). Set $f_\lambda(t) = \begin{cases} e^{-\lambda t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}$ and $g_\lambda(t) = -t f_\lambda(t)$. Then $f_\lambda, g_\lambda \in L^1(\mathbb{R})$ for all $\lambda \in \mathbb{C}_+$. The proof of the case $\mathbb{J} = \mathbb{R}_+$ can be reduced to the case $\mathbb{J} = \mathbb{R}$ noting that $\mathcal{L}_u\phi(\lambda)(s) = \check{f}_\lambda * \phi(s)$ for $s \in \mathbb{R}^+$. So, we consider the case $\mathbb{J} = \mathbb{R}$. For $\text{Re } \lambda > 0$, we have $\mathcal{L}_u\phi(\lambda)(s) = \mathcal{L}\phi_s(\lambda) = \mathcal{L}^+\phi_s(\lambda)$ and $\mathcal{L}_u\phi(\lambda)(s) = \check{f}_\lambda * \phi(s)$. This implies $\mathcal{L}_u\phi(\lambda) \in BUC(\mathbb{R}, X)$ (see [4, Proposition 1.3.2 (c)]). Now, let $(\lambda_n) \subset \mathbb{C}$, $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$, $\lambda_n \neq \lambda$ and $\text{Re } \lambda_n \geq \text{Re } \lambda/2$. One can show that for $t \geq 0$, one has $|\frac{e^{-\lambda_n t} - e^{-\lambda t}}{\lambda_n - \lambda}| \leq t e^{-(1/2)\text{Re } \lambda t}$, and so by the Lebesgue dominating convergence theorem $\|\frac{f_{\lambda_n} - f_\lambda}{\lambda_n - \lambda} - g_\lambda\|_{L^1} \rightarrow 0$ as $n \rightarrow \infty$. This implies $\|\frac{\mathcal{L}_u\phi(\lambda_n) - \mathcal{L}_u\phi(\lambda)}{\lambda_n - \lambda} - \check{g}_\lambda * \phi\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Hence $\mathcal{L}_u\phi$ is differentiable at λ and $\frac{d\mathcal{L}_u\phi(\lambda)}{d\lambda} = \check{g}_\lambda * \phi$. So, $\mathcal{L}_u\phi$

is holomorphic on \mathbb{C}_+ . If $\phi \in BUC(\mathbb{R}, X)$, then $S(\cdot)$ is strongly continuous, and so $\mathcal{L}^+\phi_s(\lambda) = \int_0^\infty e^{-\lambda t} \phi(s+t) dt = \int_0^\infty e^{-\lambda t} S(t)\phi(s) dt = (\int_0^\infty e^{-\lambda t} S(t)\phi dt)(s)$. The last equality follows by [32, Corollary 2, p. 134] since evaluation at s is a bounded linear operator. This proves (i) and (ii) on \mathbb{C}_+ because $\mathcal{C}_u\phi(\lambda)(s) = \mathcal{L}^+\phi_s(\lambda)$. The case $\operatorname{Re}\lambda < 0$ being similar, one gets (ii).

(iii) If $\lambda \in V \cap \mathbb{C}_+$, then $\overline{\mathcal{L}}_u\phi(\lambda) = \overline{\mathcal{L}}_u\phi(\lambda) \in BUC(\mathbb{R}_+, X)$ by part (i). If $\lambda = i\omega \in V \cap i\mathbb{R}$, then $\overline{\mathcal{L}}_u\phi(i\omega)(s) = e^{i\omega s}(\overline{\mathcal{L}}\phi(i\omega) - \int_0^s e^{-i\omega t}\phi(t) dt)$ is bounded uniformly continuous by Theorem 2.4 (iv). This implies $\overline{\mathcal{L}}_u\phi(i\omega) \in BUC(\mathbb{R}_+, X)$. If $\lambda \in V \cap \mathbb{C}_-$, then $\overline{\mathcal{L}}_u\phi(\lambda)(s) = e^{\lambda s}\overline{\mathcal{L}}\phi(\lambda) - \int_0^s e^{\lambda(s-t)}\phi(t) dt = e^{\lambda s}\overline{\mathcal{L}}\phi(\lambda) - h_\lambda * \overline{\phi}(s)$, where $h_\lambda(t) = e^{\lambda t}$ if $t \geq 0$ and $h_\lambda(t) = 0$ if $t < 0$ and $\overline{\phi}$ is as in the proof above. Since $\operatorname{Re}\lambda < 0$, $h_\lambda \in L^1(\mathbb{R})$ and $e^{\lambda s}$ is uniformly continuous and bounded on \mathbb{R}_+ . This implies $\overline{\mathcal{L}}_u\phi(\lambda) \in BUC(\mathbb{R}_+, X)$ for each $\lambda \in V$.

If $\lambda \in V \cap \mathbb{C} \setminus i\mathbb{R}$, then $\overline{\mathcal{C}}_u\phi(\lambda) = \mathcal{C}_u\phi(\lambda) \in BUC(\mathbb{R}, X)$ by part (ii). If $\lambda = i\omega \in V \cap i\mathbb{R}$, then $\overline{\mathcal{C}}_u\phi(i\omega)(s) = e^{i\omega s}(\overline{\mathcal{C}}\phi(i\omega) - \int_0^s e^{-i\omega t}\phi(t) dt)$ is uniformly continuous and bounded by [8, Corollary 4.4]. This proves $\overline{\mathcal{C}}_u\phi(\lambda) \in BUC(\mathbb{R}, X)$ for each $\lambda \in V$. \blacksquare

If $\phi \in L^\infty(\mathbb{J}, X)$ then $\omega \in \mathbb{R}$ is said to be *L-uniformly regular* (respectively *C-uniformly regular*) for ϕ if $i\omega$ is regular for $\mathcal{L}_u\phi : \mathbb{C}_+ \rightarrow BUC(\mathbb{R}_+, X)$ (respectively $\mathcal{C}_u\phi : \mathbb{C} \setminus i\mathbb{R} \rightarrow BUC(\mathbb{R}, X)$). The corresponding *uniform spectra* are the sets $sp^{\mathcal{L}_u}(\phi)$ and $sp^{\mathcal{C}_u}(\phi)$ of real numbers which are not L- and C-uniformly regular respectively.

Since $\mathcal{L}_u\phi(\lambda)(0) = \mathcal{L}\phi(\lambda)$, it follows

$$(4.4) \quad sp^{\mathcal{L}}(\phi) \subset sp^{\mathcal{L}_u}\phi \quad \text{and} \quad sp^{\mathcal{C}}(\phi) \subset sp^{\mathcal{C}_u}\phi.$$

Proposition 4.2. *Let $\phi \in L^\infty(\mathbb{J}, X)$. Then $sp^{\mathcal{L}}(\phi) = sp^{\mathcal{L}_u}(\phi)$ for $\mathbb{J} = \mathbb{R}_+$ and $sp^{\mathcal{C}}(\phi) = sp^{\mathcal{C}_u}(\phi)$ for $\mathbb{J} = \mathbb{R}$.*

Proof. By (4.4) we need to prove $\omega_0 \notin sp^{\mathcal{L}}(\phi)$ (respectively $\omega_0 \notin sp^{\mathcal{C}}(\phi)$) implies $\omega_0 \notin sp^{\mathcal{L}_u}(\phi)$ (respectively $\omega_0 \notin sp^{\mathcal{C}_u}(\phi)$). By Lemma 4.1 (iii), $\overline{\mathcal{L}}_u\phi(\lambda) \in BUC(\mathbb{R}_+, X)$ (respectively $\overline{\mathcal{C}}_u\phi(\lambda) \in BUC(\mathbb{R}, X)$) for each $\lambda \in V$. Moreover, $\overline{\mathcal{L}}_u\phi(\cdot)(s)$ (respectively $\overline{\mathcal{C}}_u\phi(\cdot)(s)$) is holomorphic on V for each $s \in \mathbb{R}^+$ (respectively $s \in \mathbb{R}$). By (4.3), this implies $x^* \circ (\overline{\mathcal{L}}_u\phi(\cdot)(s)) = \overline{\mathcal{L}}(x^* \circ \phi_s)$ (respectively

$x^* \circ (\overline{\mathcal{L}}_u \phi(\cdot)(s)) = \overline{\mathcal{L}}(x^* \circ \phi_s)$ is holomorphic on V for each $s \in \mathbb{R}^+$ (respectively $s \in \mathbb{R}$) and each $x^* \in X^*$. This implies $\overline{\mathcal{L}}_u \phi$ ($\overline{\mathcal{L}}_u \phi$) is holomorphic on V by [19, Definition 3.10.1, Theorem 3.10.1] since the set of functionals $\{\phi \rightarrow x^* \circ \phi(s) : x^* \in X^*, s \in \mathbb{J}\}$ is a total subspace [16, p. 418] of $(BUC(\mathbb{J}, X))^*$. This proves $\omega_0 \notin sp^{\mathcal{L}^u}(\phi)$ ($\omega_0 \notin sp^{\mathcal{C}^u}(\phi)$). \P

Corollary 4.3. *Let $\phi \in BUC(\mathbb{J}, X)$. Then*

$$sp^{\mathcal{L}}(\phi) = sp^{\mathcal{L}}(S^{\mathbb{R}^+}(\cdot)\phi) \text{ for } \mathbb{J} = \mathbb{R}_+ \text{ and } sp^{\mathcal{C}}(\phi) = sp^{\mathcal{C}}(S^{\mathbb{R}}(\cdot)\phi) \text{ for } \mathbb{J} = \mathbb{R}.$$

Proof. By Lemma 4.1, $\mathcal{L}_u \phi = \mathcal{L}(S^{\mathbb{R}^+}(\cdot)\phi)$ and $\mathcal{C}_u \phi = \mathcal{C}(S^{\mathbb{R}}(\cdot)\phi)$. This implies $sp^{\mathcal{L}^u}(\phi) = sp^{\mathcal{L}}(S^{\mathbb{R}^+}(\cdot)\phi)$ and $sp^{\mathcal{C}^u}(\phi) = sp^{\mathcal{C}}(S^{\mathbb{R}}(\cdot)\phi)$. By Proposition 4.2, one gets $sp^{\mathcal{L}}(\phi) = sp^{\mathcal{L}}(S^{\mathbb{R}^+}(\cdot)\phi)$ and $sp^{\mathcal{C}}(\phi) = sp^{\mathcal{C}}(S^{\mathbb{R}}(\cdot)\phi)$. \P

§5 CONDITIONS FOR $sp^{\mathcal{L}} = sp^{\mathcal{C}}$

In the following we indicate a subclass of $L^\infty(\mathbb{R}, X)$ for which the half-line spectrum $sp^{\mathcal{L}}$ coincides with Carleman spectrum $sp^{\mathcal{C}}$. This class includes almost periodic, almost automorphic, Levitan almost periodic and recurrent functions (see [1], [5], [7], [13], [23]). For that let $\phi \in L^\infty(\mathbb{R}, X)$. Set

$$(5.1) \quad L(\phi) = \overline{\text{span}}\{\phi_t : t \in \mathbb{R}\} \text{ and } L^+(\phi) = L(\phi)|_{\mathbb{R}_+},$$

$$(5.2) \quad LC(\phi) = \overline{\text{span}}\{\phi * f : f \in L^1(\mathbb{R})\}, LC^+(\phi) = LC(\phi)|_{\mathbb{R}^+},$$

$$(5.3) \quad m : LC(\phi) \rightarrow LC^+(\phi), \text{ where } m(\psi) = \psi|_{\mathbb{R}_+}.$$

Note that $L(\phi)$, $LC(\phi)$ are closed translation invariant subspaces of $L^\infty(\mathbb{R}, X)$, $BUC(\mathbb{R}, X)$ respectively. Moreover, using Bochner integration if $\phi \in BUC(\mathbb{R}, X)$, then $L(\phi) = LC(\phi)$ (see for example [5, Lemma 1.2.1]).

Theorem 5.1. *Let $\phi \in L^\infty(\mathbb{R}, X)$. Assume the restriction mapping $m : LC(\phi) \rightarrow LC^+(\phi)$ is an isometric linear bijection. Then $sp^{\mathcal{L}}(\phi) = sp^{\mathcal{C}}(\phi)$.*

Proof. First, we prove the case when $\phi \in BUC(\mathbb{R}, X)$. Let $i\omega_0$ be a regular point of $\mathcal{L}S(\cdot)\phi$ and $\overline{\mathcal{L}}S(\cdot)\phi$ be a holomorphic extension to $\mathbb{C} \cup V$, where V is an open disk with center $i\omega_0$. Since $S(\cdot)\phi$ is Bochner integrable in $BUC(\mathbb{R}, X)$ it follows that $\mathcal{L}S(\cdot)\phi(\lambda) \in L(\phi)$ for each $\lambda \in \mathbb{C}_+$ and hence $\overline{\mathcal{L}}S(\cdot)\phi(i\omega) \in L(\phi)$ for each $i\omega \in V \cap i\mathbb{R}$. Using a Taylor expansion for $\overline{\mathcal{L}}S(\cdot)\phi$ about the point $i\omega_0$, one can conclude $\overline{\mathcal{L}}S(\cdot)\phi(\lambda) \in L(\phi)$ for each $\lambda \in V$. Since m is an isometric linear bijection,

we conclude $m \circ \overline{\mathcal{L}}S(\cdot)\phi$ is a holomorphic extension to V of $\mathcal{L}S^+(\cdot)\phi|_{\mathbb{R}_+}$ and hence $i\omega_0$ is a regular point of $\mathcal{L}S^+(\cdot)\phi|_{\mathbb{R}_+}$. This implies $sp^{\mathcal{L}}(S^+(\cdot)\phi|_{\mathbb{R}_+}) \subset sp^{\mathcal{L}}(S(\cdot)\phi)$. The converse can be proved similarly since $m^{-1} : LC^+(\phi) \rightarrow LC(\phi)$ is also a linear isometric mapping. So, $sp^{\mathcal{L}}(S^+(\cdot)\phi|_{\mathbb{R}_+}) = sp^{\mathcal{L}}(S(\cdot)\phi)$. Hence by Corollary 4.3 and (3.6), we conclude $sp^{\mathcal{L}}(\phi) = sp^{\mathcal{L}}(S^+(\cdot)\phi|_{\mathbb{R}_+}) = sp^{\mathcal{L}}(S(\cdot)\phi) = sp^{\mathcal{C}}(S(\cdot)\phi) = sp^{\mathcal{C}}(\phi)$.

Now, assume $\phi \in L^\infty(\mathbb{R}, X)$. Then $M_h\phi \in LC(\phi) \subset BUC(\mathbb{R}, X)$ for each $h > 0$. It follows $LC(M_h\phi)$ is a closed translation invariant subspace of $LC(\phi)$. The assumptions imply $m : LC(M_h\phi) \rightarrow LC^+(M_h\phi)$ is an isometric linear bijection. So, by the above $sp^{\mathcal{L}}(M_h\phi) = sp^{\mathcal{C}}(M_h\phi)$ and by Proposition 1.1 (ii), we conclude $sp^{\mathcal{L}}(\phi) = \cup_{h>0} sp^{\mathcal{L}}(M_h\phi) = \cup_{h>0} sp^{\mathcal{C}}(M_h\phi) = sp^{\mathcal{C}}(\phi)$. \P

In the following $AP(\mathbb{R}, X)$, $AA(\mathbb{R}, X)$, $LAP_b(\mathbb{R}, X)$, $RC_b(\mathbb{R}, X)$ will denote respectively the class of almost periodic, almost automorphic, bounded Levitan-almost periodic and continuous bounded recurrent functions.

Corollary 5.2. *Let $\phi \in \mathcal{A} \in \{AP(\mathbb{R}, X), AA(\mathbb{R}, X), LAP_b(\mathbb{R}, X), RC_b(\mathbb{R}, X)\}$. Then $sp^{\mathcal{L}}(\phi) = sp^{\mathcal{C}}(\phi)$.*

Proof. This follows from Theorem 5.1, since $LC(\phi) \subset \mathcal{A}$ and $m : LC(\phi) \rightarrow LC^+(\phi)$ is a linear isometric bijection, by [5, Theorem 2.1.9].

Remark 5.3. (i) *Let $\phi(t) = e^{it^2}$. Then $L(\phi) = \phi \cdot AP(\mathbb{R}, \mathbb{C})$ and $m : L(\phi) \rightarrow L^+(\phi)$ is a linear isometric bijection. But by Example 2.2, $sp^{\mathcal{L}}(\phi) = \emptyset$ and $sp^{\mathcal{C}}(\phi) = \mathbb{R}$. Also, note that $LC(\phi) \subset C_0(\mathbb{R}, X)$ and so $L(\phi) \cap LC(\phi) = \{0\}$.*

(ii) *If $\phi \in D(\mathbb{R}, X)$, the Banach space of distal functions [13, p. 177] or $\phi \in \mathcal{MA} \cap L^\infty(\mathbb{R}, X)$ with \mathcal{A} as in Corollary 4.6, then also, $sp^{\mathcal{L}}(\phi) = sp^{\mathcal{C}}(\phi)$.*

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School of Math. Sci., P.O. Box No. 28M, Monash University, Vic. 3800.

E-mails "bolis.basit@sci.monash.edu.au", "alan.pryde@sci.monash.edu.au".